

The $S_t^1 \times S_s^1$ -valued lightcone Gauss map of a Lorentzian surface in semi-Euclidean 4-space

Donghe Pei, Lingling Kong, Jianguo Sun and Qi Wang

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Abstract

We define the notions of $S_t^1 \times S_s^1$ -valued lightcone Gauss maps, lightcone pedal surface and Lorentzian lightcone height function of Lorentzian surface in semi-Euclidean 4-space and established the relationships between singularities of these objects and geometric invariants of the surface as applications of standard techniques of singularity theory for the Lorentzian lightcone height function.

1 Introduction

In [8, 9], S.Izumiya et al studied singularities of lightcone Gauss maps and lightlike hypersurfaces of spacelike surface in Minkowski 4-space, and established the relationships between such singularities and geometric invariants of these surfaces under the action of Lorentz group. Our aim in this paper is to develop the analogous study for Lorentzian surface in semi-Euclidean 4-space \mathbb{R}_2^4 . To do this we need to develop first the local differential geometry of Lorentzian surface in semi-Euclidean 4-space \mathbb{R}_2^4 in a similar way than the classically done surfaces in Euclidean 4-space [15]. As it was to be expected, the situation presents certain peculiarities when compared with the Euclidean case. For instance, in our case it is always possible to choose two lightlike normal directions along the Lorentzian surface a frame of its normal bundle. By using this, we define a Lorentzian invariant $\mathcal{K}_l(1, \pm 1)$ and call it the *lightlike Gauss-Kronecker curvature* of the Lorentzian surface. We introduce the notion of lightcone height function and use it to show that the $S_t^1 \times S_s^1$ -valued lightcone Gauss map has a singular point if and only if the lightlike Gauss-Kronecker curvature vanishes at such point. Moreover, we show that the $S_t^1 \times S_s^1$ -valued lightcone Gauss map is a constant map if and only if the Lorentzian surface is contained in a lightlike hyperplane, so we can view the singularities of the $S_t^1 \times S_s^1$ -valued lightcone Gauss map as an estimate of the contacts of the surface with lightlike hyperplanes.

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) | x_i \in \mathbb{R} \ (i = 1, 2, 3, 4)\}$ be a 4-dimensional vector space. For any vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ in \mathbb{R}^4 , the *pseudo scalar product* of \mathbf{x} and \mathbf{y} is

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defined to be $\langle \mathbf{x}, \mathbf{y} \rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4$. We call $(\mathbb{R}^4, \langle, \rangle)$ a *semi-Euclidean 4-space* and write \mathbb{R}_2^4 instead of $(\mathbb{R}^4, \langle, \rangle)$.

We say that a vector $\mathbf{x} \in \mathbb{R}_2^4 \setminus \{\mathbf{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0, = 0$ or < 0 respectively. The norm of the vector $\mathbf{x} \in \mathbb{R}_2^4$ is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$. For a lightlike vector $\mathbf{n} \in \mathbb{R}_2^4$ and a real number c , we define the *lightlike hyperplane* with pseudo normal \mathbf{n} by

$$LHP(\mathbf{n}, c) = \{\mathbf{x} \in \mathbb{R}_2^4 | \langle \mathbf{x}, \mathbf{n} \rangle = c\}.$$

Let $\mathbf{X} : U \rightarrow \mathbb{R}_2^4$ an immersion, where $U \subset \mathbb{R}^2$ is an open subset. We denote that $M = \mathbf{X}(U)$ and identify M and U by the immersion \mathbf{X} .

We say that M is a *Lorentzian surface* if the tangent space $T_p M$ of M is a Lorentzian surface for any point $p \in M$. In this case, the normal space $N_p M$ is a Lorentzian plane. Let $\{\mathbf{e}_3(x, y), \mathbf{e}_4(x, y); p = (x, y)\}$ be an pseudo-orthonormal frame of the tangent space $T_p M$ and $\{\mathbf{e}_1(x, y), \mathbf{e}_2(x, y); p = (x, y)\}$ a pseudo-orthonormal frame of $N_p M$, where, $\mathbf{e}_1(p), \mathbf{e}_3(p)$ are unit timelike vectors and $\mathbf{e}_2, \mathbf{e}_4$ are unit spacelike vectors.

We shall now establish the fundamental formula for a Lorentzian 2-space in \mathbb{R}_2^4 by means of similar notions to those of [9].

We can write $d\mathbf{X} = \sum_{i=1}^4 \omega_i \mathbf{e}_i$ and $d\mathbf{e}_i = \sum_{j=1}^4 \omega_{ij} \mathbf{e}_j$; $i = 1, 2, 3, 4$. where ω_i and ω_{ij} are 1-forms given by $\omega_i = \delta(\mathbf{e}_i) \langle d\mathbf{X}, \mathbf{e}_i \rangle$ and $\omega_{ij} = \delta(\mathbf{e}_j) \langle d\mathbf{e}_i, \mathbf{e}_j \rangle$,

$$\text{with} \quad \delta(\mathbf{e}_i) = \langle \mathbf{e}_i, \mathbf{e}_i \rangle = \begin{cases} 1, & i = 2, 4, \\ -1, & i = 1, 3. \end{cases}$$

We have the Codazzi type equations:

$$\begin{cases} d\omega_i = \sum_{j=1}^4 \delta(\mathbf{e}_i) \delta(\mathbf{e}_j) \omega_{ij} \wedge \omega_j; \\ d\omega_{ij} = \sum_{k=1}^4 \omega_{ik} \wedge \omega_{kj}, \end{cases} \quad (1)$$

where d is exterior derivative.

Since $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij} \delta(\mathbf{e}_j)$ (where δ_{ij} is Kronecker's delta), we get

$$\omega_{ij} = -\delta(\mathbf{e}_i) \delta(\mathbf{e}_j) \omega_{ji}. \quad (2)$$

In particular, $\omega_{ii} = 0$; $i = 1, 2, 3, 4$. It follows from the fact $\langle d\mathbf{X}, \mathbf{e}_1 \rangle = \langle d\mathbf{X}, \mathbf{e}_2 \rangle = 0$ that

$$\omega_1 = \omega_2 = 0. \quad (3)$$

Therefore we have

$$\begin{cases} 0 = d\omega_1 = \delta(\mathbf{e}_j) \sum_{j=1}^4 \omega_{1j} \wedge \omega_j = \delta(\mathbf{e}_j) \sum_{j=3}^4 \omega_{1j} \wedge \omega_j = -\omega_{13} \wedge \omega_3 + \omega_{14} \wedge \omega_4; \\ 0 = d\omega_2 = \delta(\mathbf{e}_j) \sum_{j=1}^4 \omega_{2j} \wedge \omega_j = \delta(\mathbf{e}_j) \sum_{j=3}^4 \omega_{2j} \wedge \omega_j = -\omega_{23} \wedge \omega_3 + \omega_{24} \wedge \omega_4. \end{cases} \quad (4)$$

By Cartan's lemma, we can write

$$\begin{cases} \omega_{13} = a\omega_3 + b\omega_4; & \omega_{14} = -b\omega_3 - c\omega_4; \\ \omega_{23} = \bar{a}\omega_3 + \bar{b}\omega_4; & \omega_{24} = -\bar{b}\omega_3 - \bar{c}\omega_4. \end{cases} \quad (5)$$

for appropriate functions $a, b, c, \bar{a}, \bar{b}, \bar{c}$.

Since $\langle d\mathbf{X}, \mathbf{e}_1 \rangle = \langle d\mathbf{X}, \mathbf{e}_2 \rangle = 0$,

$$\begin{aligned}
\langle d^2\mathbf{X}, \mathbf{e}_1 \rangle &= -\langle d\mathbf{X}, d\mathbf{e}_1 \rangle \\
&= -\left\langle \sum_{i=1}^4 \omega_i \mathbf{e}_i, \sum_{j=1}^4 \omega_{1j} \mathbf{e}_j \right\rangle = -\left\langle \sum_{i=3}^4 \omega_i \mathbf{e}_i, \sum_{j=2}^4 \omega_{1j} \mathbf{e}_j \right\rangle \\
&= -(-\omega_3 \omega_{13} + \omega_4 \omega_{14}) \\
&= \omega_3(a\omega_3 + b\omega_4) + \omega_4(b\omega_3 + c\omega_4) \\
&= a\omega_3^2 + 2b\omega_3\omega_4 + c\omega_4^2.
\end{aligned}$$

As the same $\langle d^2\mathbf{X}, \mathbf{e}_2 \rangle = \bar{a}\omega_3^2 + 2\bar{b}\omega_3\omega_4 + \bar{c}\omega_4^2$.

Then we have a vector-valued quadratic form

$$-\langle d^2\mathbf{X}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle d^2\mathbf{X}, \mathbf{e}_2 \rangle \mathbf{e}_2 = -(a\omega_3^2 + c\omega_4^2 + 2b\omega_3\omega_4) \mathbf{e}_1 + (\bar{a}\omega_3^2 + \bar{c}\omega_4^2 + 2\bar{b}\omega_3\omega_4) \mathbf{e}_2,$$

which is called the second fundamental form of the Lorentz surface.

By using equations (2) and a straight forward calculation leads us to the following equations:

$$d \begin{pmatrix} \mathbf{e}_1 + \mathbf{e}_2 \\ \mathbf{e}_1 - \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} + \omega_{23} & \omega_{14} + \omega_{24} \\ -\omega_{12} & 0 & \omega_{13} - \omega_{23} & \omega_{14} - \omega_{24} \\ -(\omega_{13} + \omega_{23})/2 & (\omega_{23} - \omega_{13})/2 & 0 & \omega_{34} \\ (\omega_{14} + \omega_{24})/2 & (\omega_{14} - \omega_{24})/2 & \omega_{34} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 - \mathbf{e}_2 \\ \mathbf{e}_1 + \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix}$$

On the other hand, we define

$$LC_p = \{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}_2^4 \mid -(x_1 - p_1)^2 - (x_2 - p_2)^2 + (x_3 - p_3)^2 + (x_4 - p_4)^2 = 0\}$$

and

$$S_t^1 \times S_s^1 = \{\mathbf{x} = (x_1, x_2, x_3, x_4) \in LC_0 \mid x_1^2 + x_2^2 = 1, x_1 \geq 0, x_2 \geq 0\},$$

where $p = (p_1, p_2, p_3, p_4) \in \mathbb{R}_2^4$, S_t^1 denotes the *timelike circle* and S_s^1 denotes the *spacelike circle*.

We call $LC_p^* = LC_p \setminus \{p\}$ a *lightcone* at the vertex p . Given any lightlike vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$, we have $\tilde{\mathbf{x}} = (\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, \frac{x_3}{\sqrt{x_1^2 + x_2^2}}, \frac{x_4}{\sqrt{x_1^2 + x_2^2}}) \in S_t^1 \times S_s^1$.

Let $\mathbf{e}_1 = (a_1, a_2, a_3, a_4)$, $\mathbf{e}_2 = (b_1, b_2, b_3, b_4)$, and $\xi^\pm = \sqrt{(a_1 - b_1)^2 \pm (a_2 - b_2)^2}$, then we have the following fundamental formula:

$$d \begin{pmatrix} \widetilde{\mathbf{e}_1 + \mathbf{e}_2} \\ \widetilde{\mathbf{e}_1 - \mathbf{e}_2} \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_{12} - \frac{d\xi^+}{\xi^+} & \frac{\omega_{13} + \omega_{23}}{\xi^+} & \frac{\omega_{14} + \omega_{24}}{\xi^+} \\ -\omega_{12} - \frac{d\xi^-}{\xi^-} & 0 & \frac{\omega_{13} - \omega_{23}}{\xi^-} & \frac{\omega_{14} - \omega_{24}}{\xi^-} \\ -(\omega_{13} + \omega_{23})/2 & (\omega_{23} - \omega_{13})/2 & 0 & \omega_{34} \\ (\omega_{14} + \omega_{24})/2 & (\omega_{14} - \omega_{24})/2 & \omega_{34} & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\mathbf{e}_1 - \mathbf{e}_2} \\ \widetilde{\mathbf{e}_1 + \mathbf{e}_2} \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{pmatrix}$$

Given $\mathbf{v} = x\mathbf{e}_1 + y\mathbf{e}_2 \in N_p M$, we have $d\mathbf{v} = dx\mathbf{e}_1 + x d\mathbf{e}_1 + dy\mathbf{e}_2 + y d\mathbf{e}_2$, and then

$$\langle d\mathbf{v}, \mathbf{e}_3 \rangle \wedge \langle d\mathbf{v}, \mathbf{e}_4 \rangle = \mathcal{K}_l(x, y) \omega_3 \wedge \omega_4,$$

where the function \mathcal{K}_l as follows:

$$\mathcal{K}_l(x, y) = (ax + \bar{a}y)(cx + \bar{c}y) - (bx + \bar{b}y)^2.$$

On the other hand, we define two maps

$$LG_M^\pm : U \longrightarrow S_t^1 \times S_s^1$$

by $LG_M^\pm(x, y) = \widetilde{e_1 \pm e_2}(x, y)$. Each one of these maps shall be called $S_t^1 \times S_s^1$ -valued *lightcone Gauss map* of $\mathbf{X}(U) = M$.

Now we introduce the notion of Lorentzian lightcone height functions on the Lorentzian surface in \mathbb{R}_2^4 which is useful for the study of singularities of $S_t^1 \times S_s^1$ -valued lightcone Gauss maps.

For a Lorentzian surface $M(=\mathbf{X}(U)) \in \mathbb{R}_2^4$, we now define a function

$$H : U \times S_t^1 \times S_s^1 \longrightarrow \mathbb{R}$$

by $H((x, y), \boldsymbol{\lambda}) = \langle \mathbf{X}(x, y), \boldsymbol{\lambda} \rangle$, where $\boldsymbol{\lambda} = (\cos \theta, \sin \theta, \lambda_3, \lambda_4) \in S_t^1 \times S_s^1$. We call H the *Lorentzian lightcone height function* on the surface M . We denote that $h_{\lambda_0}(x, y) = H(x, y, \boldsymbol{\lambda}_0)$, for any fixed $\boldsymbol{\lambda}_0 \in S_t^1 \times S_s^1$. Then we have the following proposition.

Proposition 1.1 Let M be a Lorentzian surface in \mathbb{R}_2^4 and $H : U \times S_t^1 \times S_s^1 \longrightarrow \mathbb{R}$ a Lorentzian lightcone height function. Then we have the following assertions:

(1) $(\partial h_\lambda / \partial x)(p_0) = (\partial h_\lambda / \partial y)(p_0) = 0$ if and only if

$$\boldsymbol{\lambda} = \mu(e_1 \pm e_2)(p_0) = \widetilde{e_1 \pm e_2}(p_0),$$

where $e_1(p_0) = (a_1, a_2, a_3, a_4)$, $e_2(p_0) = (b_1, b_2, b_3, b_4)$ and $\mu = \frac{1}{\sqrt{(a_1 \pm b_1)^2 + (a_2 \pm b_2)^2}}$; for any point $p_0 = (x_0, y_0) \in M$,

(2) $(\partial h_\lambda / \partial x)(p_0) = (\partial h_\lambda / \partial y)(p_0) = \det \mathcal{H}(h_\lambda)(p_0) = 0$ if and only if

$$\boldsymbol{\lambda} = \widetilde{e_1 \pm e_2}(p_0) \text{ and } \mathcal{K}_l(1, \pm 1)(p_0) = 0.$$

Here, $\det \mathcal{H}(h_\lambda)(x, y)$ is the determinant of the Hessian matrix of h_λ at (x, y) .

Proof. By a straight forward calculation, $(\partial h_\lambda / \partial x)(p_0) = (\partial h_\lambda / \partial y)(p_0) = 0$ if and only if

$$\langle \mathbf{X}_x, \boldsymbol{\lambda} \rangle(p_0) = \langle \mathbf{X}_y, \boldsymbol{\lambda} \rangle(p_0) = 0.$$

It is equivalent to the condition that $\boldsymbol{\lambda} \in N_{p_0}M$ and $\boldsymbol{\lambda} \in S_t^1 \times S_s^1$. This means that $\boldsymbol{\lambda} = \mu(e_1 \pm e_2) = \widetilde{e_1 \pm e_2}$.

On the other hand, we now choose local coordinates such that \mathbf{X} is given by the Monge form $\mathbf{X}(x, y) = (f_1(x, y), x, f_2(x, y), y)$ and $e_1(p_0) = (1, 0, 0, 0)$ and $e_2(p_0) = (0, 0, 1, 0)$. Since

$$\det \mathcal{H}(h_\lambda)(x, y) = \begin{vmatrix} \langle \mathbf{X}_{xx}, \boldsymbol{\lambda} \rangle & \langle \mathbf{X}_{xy}, \boldsymbol{\lambda} \rangle \\ \langle \mathbf{X}_{xy}, \boldsymbol{\lambda} \rangle & \langle \mathbf{X}_{yy}, \boldsymbol{\lambda} \rangle \end{vmatrix} = 0$$

and $\boldsymbol{\lambda}(p_0) = (1, 0, \pm 1, 0)$, we have

$$\begin{aligned} & \left| \begin{matrix} \langle (f_{1xx}, 0, f_{2xx}, 0), \boldsymbol{\lambda}(p_0) \rangle & \langle (f_{1xy}, 0, f_{2xy}, 0), \boldsymbol{\lambda}(p_0) \rangle \\ \langle (f_{1xy}, 0, f_{2xy}, 0), \boldsymbol{\lambda}(p_0) \rangle & \langle (f_{1yy}, 0, f_{2yy}, 0), \boldsymbol{\lambda}(p_0) \rangle \end{matrix} \right| \\ &= \begin{vmatrix} a \pm \bar{a} & b \pm \bar{b} \\ b \pm \bar{b} & c \pm \bar{c} \end{vmatrix} = 0. \end{aligned}$$

This is equivalent to the condition that $\mathcal{K}_l(1, \pm 1)(x, y) = 0$ and $\boldsymbol{\lambda}(p_0) = \widetilde{e_1 \pm e_2}$. \square

As a corollary of the above proposition, we have the following theorem.

Theorem 1.2 Under the same assumption as the assumption of the above proposition, the following conditions are equivalent:

- (1) $p \in M$ is a degenerate singular point of Lorentzian lightcone height function h_λ .
- (2) There is $\lambda \in S_t^1 \times S_s^1$ such that (p, λ) is a singular point of the $S_t^1 \times S_s^1$ -valued lightcone Gauss map LG_M^\pm .
- (3) $\mathcal{K}_l(1, \pm 1)(p) = 0$.

Proof. We denote that

$$\Sigma(H) = \left\{ (p, \lambda) \in U \times S_t^1 \times S_s^1 \mid \frac{\partial h_\lambda}{\partial x}(p) = \frac{\partial h_\lambda}{\partial y}(p) = 0 \right\}.$$

By above proposition, (1), we have

$$\Sigma(H) = \{ (p, \lambda) \in U \times S_t^1 \times S_s^1 \mid \lambda = \widetilde{e_1 \pm e_2}(p) \}.$$

We now consider the canonical projection $\pi : U \times S_t^1 \times S_s^1 \longrightarrow S_t^1 \times S_s^1$, then $\pi|\Sigma(H)$ can be identified to the $S_t^1 \times S_s^1$ -valued lightcone Gauss map LG_M^\pm . Under this identification, we can show that the condition (1) is equivalent to the condition (2).

Above proposition, (2) means that the condition (2) is equivalent to the condition (3).

Theorem 1.3 Let M be a Lorentzian surface in \mathbb{R}_2^4 .

- (1) The $S_t^1 \times S_s^1$ -valued lightcone Gauss maps LG_M^+ (respectively, LG_M^-) is constant if and only if there exists a unique lightlike hyperplane $LHP(\mathbf{v}^+, c^+)$ (respectively, $LHP(\mathbf{v}^-, c^-)$) such that $M \subset LHP(\mathbf{v}^+, c^+)$ (respectively, $M \subset LHP(\mathbf{v}^-, c^-)$), where $\mathbf{v}^\pm = \widetilde{e_1 \pm e_2}(x, y)$ and $\langle \mathbf{X}(x, y), \mathbf{v}^\pm \rangle = c^\pm$ for any $(x, y) \in M$.
- (2) Both of the $S_t^1 \times S_s^1$ -valued lightcone Gauss maps LG_M^+ and LG_M^- are constant if and only if M is a Lorentzian 2-plane. In this case, the intersection of lightlike hyperplanes

$$LHP(\widetilde{e_1 + e_2}, c^+) \cap LHP(\widetilde{e_1 - e_2}, c^-)$$

is the Lorentzian 2-plane M .

Proof. (1) For convenience, we consider the case when $LG_M^+(x, y) = \widetilde{e_1 + e_2}(x, y)$ is constant, so that we have

$$d\langle \mathbf{X}, \widetilde{e_1 + e_2} \rangle = \langle d\mathbf{X}, \widetilde{e_1 + e_2} \rangle + \langle \mathbf{X}, d\widetilde{e_1 + e_2} \rangle = 0.$$

Therefore, $\langle \mathbf{X}, \widetilde{e_1 + e_2} \rangle \equiv c^+$. This means that $M = \mathbf{X}(U) \subset LHP(\mathbf{v}^+, c^+)$, where $\mathbf{v}^+ = \widetilde{e_1 + e_2}(x, y)$. For the converse assertion, suppose that there exists a lightlike vector \mathbf{v} and a real number c such that $\mathbf{X}(U) = M \subset LHP(\mathbf{v}, c)$. Since $\langle \mathbf{X}(x, y), \mathbf{v} \rangle = c$, we have $d\langle \mathbf{X}(x, y), \mathbf{v} \rangle = 0$. This means that \mathbf{v} is a lightlike normal vector of M . Thus we have $\widetilde{\mathbf{v}} = \widetilde{e_1 \pm e_2}(x, y)$. This completes the proof of the assertion (1).

Since $\mathbf{v}^+ \notin LHP(\mathbf{v}^-, c^-)$ and $\mathbf{v}^- \notin LHP(\mathbf{v}^+, c^+)$, $LHP(\mathbf{v}^-, c^-)$ and $LHP(\mathbf{v}^+, c^+)$ intersect transversally. By the assertion (1), both of the $S_t^1 \times S_s^1$ -valued lightcone Gauss maps LG_M^+ and LG_M^- are constant if and only if $M \subset LHP(\mathbf{v}^+, c^+) \cap LHP(\mathbf{v}^-, c^-)$. Here, the intersection is a Lorentzian 2-plane. Thus we have the assertion (2). \square

We say that a point $p_0 = (x_0, y_0)$ is a *Lorentzian lightlike parabolic point* of M if $\mathcal{K}_l(1, 1)(p_0) = 0$ or $\mathcal{K}_l(1, -1)(p_0) = 0$.

2 The lightcone pedal surface

In this section we consider a singular hyperplane in the lightcone LC_0 associated to M whose singularities correspond to singularities of the $S_t^1 \times S_s^2$ -valued lightcone Gauss map of M . We now define a family of functions

$$\tilde{H} : U \times LC_0 \longrightarrow \mathbb{R}$$

by

$$\tilde{H}((x, y), \mathbf{v}) = \langle \mathbf{X}(x, y), \tilde{\mathbf{v}} \rangle - \sqrt{v_1^2 + v_2^2},$$

where $\mathbf{v} = (v_1, v_2, v_3, v_4)$. We call \tilde{H} the *extended Lorentzian lightcone height function* of $M = \mathbf{X}(U)$. As a corollary of above proposition, we have the following proposition.

Proposition 2.1 Let M be a Lorentzian surface in \mathbb{R}_2^4 and $\tilde{H} : M \times LC_0 \longrightarrow \mathbb{R}$ the extended Lorentzian lightcone height function of M . For $p_0 = (x_0, y_0)$ and $\mathbf{v}_0 \in LC_0$, we have the following:

(1) $\tilde{H}(p_0, \mathbf{v}_0) = (\partial\tilde{H}/\partial x)(p_0, \mathbf{v}_0) = (\partial\tilde{H}/\partial y)(p_0, \mathbf{v}_0) = 0$ if and only if

$$\tilde{\mathbf{v}}_0 = \widetilde{\mathbf{e}_1 \pm \mathbf{e}_2}(p_0) \text{ and } \sqrt{v_1^2 + v_2^2} = \langle \mathbf{X}(p_0), \widetilde{\mathbf{e}_1 \pm \mathbf{e}_2}(p_0) \rangle.$$

(2)

$$\tilde{H}(p_0, \mathbf{v}_0) = \frac{\partial\tilde{H}}{\partial x}(p_0, \mathbf{v}_0) = \frac{\partial\tilde{H}}{\partial y}(p_0, \mathbf{v}_0) = \det\mathcal{H}(\tilde{h}_v)(p_0) = 0$$

if and only if

$$\tilde{\mathbf{v}}_0 = \widetilde{\mathbf{e}_1 \pm \mathbf{e}_2}(p_0), \sqrt{v_1^2 + v_2^2} = \langle \mathbf{X}(p_0), \widetilde{\mathbf{e}_1 \pm \mathbf{e}_2}(p_0) \rangle \text{ and } \mathcal{K}_l(1, \pm 1)(p_0) = 0.$$

Here, for a fixed $\mathbf{v} \in LC_0$, $\tilde{H}((x, y), \mathbf{v}) = \tilde{h}_v(x, y)$.

The assertion of proposition 2.1 means that the discriminant set of the extended Lorentzian lightcone height function \tilde{H} is given by

$$\mathcal{D}_{\tilde{H}} = \left\{ \mathbf{v} \mid \mathbf{v} = \langle \mathbf{X}(x, y), \widetilde{\mathbf{e}_1 \pm \mathbf{e}_2}(x, y) \rangle (\widetilde{\mathbf{e}_1 \pm \mathbf{e}_2})(x, y) \text{ for some } (x, y) \in U \right\}.$$

Therefore we now define a pair of singular surface in LC_0 by

$$LP_M^\pm(p) = LP_M^\pm(x, y) = \langle \mathbf{X}(x, y), \widetilde{\mathbf{e}_1 \pm \mathbf{e}_2}(x, y) \rangle (\widetilde{\mathbf{e}_1 \pm \mathbf{e}_2})(x, y).$$

We call each LP^\pm the *lightcone pedal surface* of $\mathbf{X}(U) = M$. A singularity of the lightcone pedal surface exactly corresponds to a singularity of the $S_t^1 \times S_s^2$ -valued lightcone Gauss map.

We define a pair of hyperplane $LH_M^\pm : M \times \mathbb{R} \rightarrow \mathbb{R}_2^4$ by

$$LH_M^\pm(p, u) = LH_M^\pm(x, y, u) = X(x, y) + u(\widetilde{\mathbf{e}_1 \pm \mathbf{e}_2})(x, y),$$

where $p = X(x, y)$ we call LH_M^\pm the lightlike hyperplane along M .

We now explain the reason why such a correspondence exists from the view point of Symplectic and Contact geometry. We consider a point $\mathbf{v} = (v_1, v_2, v_3, v_4) \in LC_0$, then we have a relation $v_1 = \sqrt{-v_2^2 + v_3^2 + v_4^2}$. We adopt the coordinate (v_2, v_3, v_4) of the manifold LC_0 . We now consider

the projective cotangent bundle $\pi : PT^*(LC_0) \longrightarrow LC_0$ with the canonical contact structure. We review geometric properties of this space. Consider the tangent bundle $\tau : TPT^*(LC_0) \rightarrow PT^*(LC_0)$ and the differential map $d\pi : TPT^*(LC_0) \rightarrow TLC_0$ of π . For any $X \in TPT^*(LC_0)$, there exists an element $\alpha \in T^*(LC_0)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(LC_0)$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(LC_0)$ by

$$K = \{X \in TPT^*(LC_0) | \tau(X)(d\pi(X)) = 0\}.$$

Since we consider the coordinate (v_2, v_3, v_4) , we have the trivialization $PT^*(LC_0) \cong LC_0 \times P(\mathbb{R}^2)^*$, we call

$$((v_2, v_3, v_4), [\xi_2 : \xi_3 : \xi_4])$$

a *homogeneous coordinate*, where $[\xi_2 : \xi_3 : \xi_4]$ is the homogeneous coordinate of the dual projective space $P(\mathbb{R}^2)^*$.

It is easy to show that $X \in K_{(x, [\xi])}$ if and only if $\sum_{i=2}^4 \mu_i \xi_i = 0$, where $d\pi(X) = \sum_{i=2}^4 \mu_i \frac{\partial}{\partial v_i}$. An immersion $i : L \rightarrow PT^*(LC_0)$ is said to be a *Legendrian immersion* if $\dim L = 2$ and $di_q(T_q L) \subset K_{i(q)}$ for any $q \in L$. We also call the map $\pi \circ i$ the *Legendrian map* and the set $W(i) = \text{image } \pi \circ i$ the *wave front* of i . Moreover, i (or, the image of i) is called the *Legendrian lift* of $W(i)$.

In order to study the lightcone pedal surface, we give a quick survey on the Legendrian singularity theory mainly due to Arnol'd-Zakalyukin [1, 19]. Although the general theory has been described for general dimension, we only consider the 3-dimensional case for the purpose. Let $F : (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that F is a *Morse family* if the mapping

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is non-singular, where $(q, x) = (q_1, \dots, q_k, x_1, x_2, x_3) \in (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0})$. In this case we have a smooth 2-dimensional submanifold

$$\Sigma_*(F) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \mid F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \dots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}$$

and the map germ $\Phi_F : (\Sigma_*(F), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^3$ defined by

$$\Phi_F(q, x) = \left(x, \left[\frac{\partial F}{\partial x_1}(q, x) : \frac{\partial F}{\partial x_2}(q, x) : \frac{\partial F}{\partial x_3}(q, x) \right] \right)$$

is a Legendrian immersion. Then we have the following fundamental theorem of Arnol'd-Zakalyukin [1, 19].

Proposition 2.2 All Legendrian submanifold germs in $PT^*\mathbb{R}^3$ are constructed by the above method.

We call F a *generating family* of Φ_F . Therefore the corresponding wave front is

$$W(\Phi_F) = \left\{ x \in \mathbb{R}^3 \mid \text{there exists } q \in \mathbb{R}^k \text{ such that } F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \dots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}.$$

By definition, we have $\mathcal{D}_F = W(\Phi_F)$. By the previous arguments, the lightcone pedal surface LP_M^\pm is the discriminant set of the extended Lorentzian lightcone height function \tilde{H} . We have the following proposition.

Proposition 2.3 The extended Lorentzian lightcone height function \tilde{H} is a Morse family.

Proof. We define another family of function

$$\bar{H} : U \times S_t^1 \times S_s^2 \times \mathbb{R} \longrightarrow \mathbb{R}$$

by $\bar{H}((x, y), \mathbf{w}, r) = \langle \mathbf{X}(x, y), \mathbf{w} \rangle - r$. We consider a C^∞ -diffeomorphism

$$\Phi : U \times S_t^1 \times S_s^2 \times \mathbb{R} \longrightarrow LC_0$$

defined by $\Phi((x, y), \mathbf{w}, r) = ((x, y), r\mathbf{w})$. Then we have $\tilde{H} = \bar{H} \circ \Phi$. It is enough to show that \bar{H} is a Morse family. For any $\mathbf{w} = (\cos \theta, \sin \theta, w_3, w_4) \in S_t^1 \times S_s^2$, we have $w_3 = \sqrt{1 - w_4^2}$, so that

$$\bar{H}((x, y), \mathbf{w}, r) = -x_1(p) \cos \theta - x_2(p) \sin \theta + x_3(p) \sqrt{1 - w_4^2} + x_4(p) w_4 - r,$$

where $\mathbf{X}(x, y) = \mathbf{X}(p) = (x_1(p), x_2(p), x_3(p), x_4(p))$. We now prove that the mapping

$$\Delta^* \bar{H} = \left(\bar{H}, \frac{\partial \bar{H}}{\partial x}, \frac{\partial \bar{H}}{\partial y} \right)$$

is non-singular at $w \in \mathcal{D}_{\bar{H}}$. The Jacobian matrix of $\Delta^* \bar{H}$ is given as follows:

$$\begin{pmatrix} \langle \mathbf{X}_x, \mathbf{w} \rangle & \langle \mathbf{X}_y, \mathbf{w} \rangle & x_1 \sin \theta - x_2 \cos \theta & -x_3 \frac{w_4}{w_3} + x_4 & -1 \\ \langle \mathbf{X}_{xx}, \mathbf{w} \rangle & \langle \mathbf{X}_{xy}, \mathbf{w} \rangle & x_{1,x} \sin \theta - x_{2,x} \cos \theta & -x_{3,x} \frac{w_4}{w_3} + x_{4,x} & 0 \\ \langle \mathbf{X}_{xy}, \mathbf{w} \rangle & \langle \mathbf{X}_{yy}, \mathbf{w} \rangle & x_{1,y} \sin \theta - x_{2,y} \cos \theta & -x_{3,y} \frac{w_4}{w_3} + x_{4,y} & 0 \end{pmatrix}.$$

By a straight forward calculation, the determinant of the matrix

$$A = \begin{pmatrix} x_{1,x} \sin \theta - x_{2,x} \cos \theta & -x_{3,x} \frac{w_4}{w_3} + x_{4,x} \\ x_{1,y} \sin \theta - x_{2,y} \cos \theta & -x_{3,y} \frac{w_4}{w_3} + x_{4,y} \end{pmatrix}$$

is equal to

$$\frac{1}{2w_3} \begin{vmatrix} \sin \theta & \cos \theta & -w_3 & w_4 \\ \sin \theta & -\cos \theta & -w_3 & w_4 \\ x_{1,x} & x_{2,x} & x_{3,x} & x_{4,x} \\ x_{1,y} & x_{2,y} & x_{3,y} & x_{4,y} \end{vmatrix}.$$

If $\det A = 0$ then $(\cos \theta, \sin \theta, -w_3, w_4) \in T_p M$. So that $(\cos \theta, \sin \theta, -w_1, w_2) \in N_p M \cap S_t^1 \times S_s^2$. It is impossible because $w = (\cos \theta, \sin \theta, w_1, w_2) \in N_p M \cap S_t^1 \times S_s^2$ and $N_p M$ is a Lorentzian 2-plane. Hence $\det A \neq 0$. \square

By proposition 2.3, we remark that the lightcone pedal surface LP_M^\pm are wave fronts and the extended Lorentzian lightcone height function \tilde{H} gives generating families of the Legendrian lifts of LP_M^\pm .

3 Contact with lightlike hyperplanes

In this section we consider the geometric meanings of the singularities of the $S_t^1 \times S_s^2$ -valued lightcone Gauss map (respectively, the lightcone pedal surface) of $\mathbf{X}(U) = M$. We consider the contact between Lorentzian surface and lightlike hyperplane like as the classical differential geometry. In the first place, we briefly review the theory of contact due to Montaldi [20]. Let X_i, Y_i ($i = 1, 2$) be submanifolds of \mathbb{R}^n with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that *the contact of X_1 and Y_1 at y_1 is same type as the contact of X_2 and Y_2 at y_2* if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \longrightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. It is clear that in the definition \mathbb{R}^n could be replaced by any manifold. In his paper [20], Montaldi gives a characterization of the notion of contact by using the terminology of singularity theory.

Theorem 3.1 *Let X_i, Y_i ($i = 1, 2$) be submanifolds of \mathbb{R}^n with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i : (X_i, x_i) \longrightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \longrightarrow (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then*

$$K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$$

if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are \mathcal{K} -equivalent.

We now consider a function $\mathcal{H} : \mathbb{R}_2^4 \times LC_0 \longrightarrow \mathbb{R}$ defined by $\mathcal{H}(\mathbf{x}, \mathbf{v}) = \langle \mathbf{x}, \tilde{\mathbf{v}} \rangle - \sqrt{v_1^2 + v_2^2}$. For any $\mathbf{v}_0 \in LC_0$, we denote that $\mathfrak{h}_{\mathbf{v}_0}(\mathbf{x}) = \mathcal{H}(\mathbf{x}, \mathbf{v}_0)$ and we have a lightlike hyperplane $\mathfrak{h}_{\mathbf{v}_0}^{-1}(0) = LHP(\tilde{\mathbf{v}}_0, \sqrt{v_{0,1}^2 + v_{0,2}^2})$. For any $p_0 = (x_0, y_0) \in U$, we consider the lightlike vector $\mathbf{v}_0^\pm = \mathbf{e}_1 \pm \mathbf{e}_2(p_0)$ and $c^\pm = \langle \mathbf{X}(p_0), \tilde{\mathbf{v}}_0^\pm \rangle$, then we have

$$\mathfrak{h}_{\mathbf{v}_0^\pm} \circ \mathbf{X}(p_0) = \mathcal{H} \circ (\mathbf{X} \times id_{LC_0})(p_0, \mathbf{v}_0^\pm) = H((x_0, y_0), \tilde{\mathbf{v}}_0^\pm) - c^\pm = 0.$$

We also have relations that

$$\frac{\partial \mathfrak{h}_{\mathbf{v}_0^\pm} \circ \mathbf{X}}{\partial x}(p_0) = \frac{\partial H}{\partial x}(p_0, \mathbf{v}_0^\pm) = 0,$$

and

$$\frac{\partial \mathfrak{h}_{\mathbf{v}_0^\pm} \circ \mathbf{X}}{\partial y}(p_0) = \frac{\partial H}{\partial y}(p_0, \mathbf{v}_0^\pm) = 0$$

This means that the lightlike hyperplane $\mathfrak{h}_{\mathbf{v}_0^\pm}^{-1}(0) = LHP(\tilde{\mathbf{v}}_0^\pm, c^\pm)$ is tangent to $M = \mathbf{X}(U)$ at p_0 . In this case, we call each $LHP(\tilde{\mathbf{v}}_0^\pm, c^\pm)$ the *tangent lightlike hyperplane* of $M = \mathbf{X}(U)$ at $p_0 = \mathbf{X}(x_0, y_0)$. Moreover, the intersection

$$LHP(\tilde{\mathbf{v}}_0^+, c^+) \cap LHP(\tilde{\mathbf{v}}_0^-, c^-)$$

is the tangent plane of M at p_0 . Let $\mathbf{v}_1, \mathbf{v}_2$ be lightlike vectors. If $\mathbf{v}_1, \mathbf{v}_2$ are linearly dependent, then corresponding lightlike hyperplanes $LHP(\mathbf{v}_1, c_1)$ and $LHP(\mathbf{v}_2, c_2)$ are parallel. Then we have the following simple lemma.

Lemma 3.2 Let $\mathbf{X} : U \longrightarrow \mathbb{R}_2^4$ be an immersion with $\mathbf{X}(U)$ is a Lorentzian surface and $\sigma = \pm$. Consider two points $p_1 = \mathbf{X}(x_1, y_1), p_2 = \mathbf{X}(x_2, y_2)$. Then we have the following assertions:

- (1) $LG_M^\sigma(p_1) = LG_M^\sigma(p_2)$ if and only if $LHP(\mathbf{v}_1^\sigma, c_1^\sigma)$ and $LHP(\mathbf{v}_2^\sigma, c_2^\sigma)$ are parallel.
- (2) $LP_M^\sigma(p_1) = LP_M^\sigma(p_2)$ if and only if $LHP(\mathbf{v}_1^\sigma, c_1^\sigma) = LHP(\mathbf{v}_2^\sigma, c_2^\sigma)$.

Here, $\mathbf{v}_i^\pm = \widetilde{\mathbf{e}_1 \pm \mathbf{e}_2(p_i)}$ and $c_i^\pm = \langle \mathbf{X}(x_i, y_i), \mathbf{v}_i^\pm \rangle$ for $i = 1, 2$.

On the other hand, for any map $f : N \longrightarrow P$, we denote $\Sigma(f)$ the set of singular points of f and $D(f) = f(\Sigma(f))$. In this case we call $f|_{\Sigma(f)} : \Sigma(f) \longrightarrow D(f)$ the *critical part* of the mapping f . For any Morse family $F : (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$, $(F^{-1}(0), \mathbf{0})$ is a smooth hypersurface, so that we define a smooth map germ $\pi_F : (F^{-1}(0), \mathbf{0}) \longrightarrow (\mathbb{R}^3, 0)$ by $\pi_F(q, x) = x$. We can easily show that $\Sigma_*(F) = \Sigma(\pi_F)$. Therefore, the corresponding Legendrian map $\pi \circ \Phi_F$ is the critical part of π_F .

We now introduce an equivalence relation among Legendrian immersion germs. Let $i : (L, p) \subset (PT^*\mathbb{R}^3, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^3, p')$ be Legendrian immersion germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^3, p) \longrightarrow (PT^*\mathbb{R}^3, p')$ such that H preserves fibers of π and that $H(L) = L'$. A Legendrian immersion germ into $PT^*\mathbb{R}^3$ at a point is said to be Legendrian stable if for every map with the given germ there is a neighbourhood in the space of Legendrian immersions (in the Whitney C^∞ topology) and a neighbourhood of the original point such that each Legendrian immersion belonging to the first neighbourhood has in the second neighbourhood a point at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift $i : (L, p) \subset (PT^*\mathbb{R}^3, p)$ is uniquely determined on the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs:

Proposition 3.3 Let $i : (L, p) \subset (PT^*\mathbb{R}^3, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^3, p')$ be Legendrian immersion germs such that regular sets of $\pi \circ i, \pi \circ i'$ are dense respectively. Then i, i' are Legendrian equivalent if and only if wave front sets $W(i), W(i')$ are diffeomorphic as set germs.

This result has been firstly pointed out by Zakalyukin [27]. The assumption in the above proposition is a generic condition for i, i' . Especially, if i, i' are Legendrian stable, then these satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote \mathcal{E}_n the local ring of function germs $(\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\}$.

Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be function germs. We say that F and G are *P-K-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back \mathbb{R} -algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.

Let $F : (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ a function germ. We say that F is a *K-versal deformation* of $f = F|_{\mathbb{R}^k \times \{0\}}$ if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1} \Big|_{\mathbb{R}^k \times \{\mathbf{0}\}}, \frac{\partial F}{\partial x_2} \Big|_{\mathbb{R}^k \times \{\mathbf{0}\}}, \frac{\partial F}{\partial x_3} \Big|_{\mathbb{R}^k \times \{\mathbf{0}\}} \right\rangle_{\mathbb{R}}$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}.$$

(See [9].)

The main result in Arnol'd-Zakalyukin's theory [1, 19] is as follows:

Theorem 3.4 Let $F, G : (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families. Then

- (1) Φ_F and Φ_G are Legendrian equivalent if and only if F, G are *P-K-equivalent*,
- (2) Φ_F is Legendrian stable if and only if F is a *K-versal deformation* of $F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$.

Since F, G are function germs on the common space germ $(\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0})$, we do not need the notion of stably P - \mathcal{K} -equivalences under this situation. By the uniqueness result of the \mathcal{K} -versal deformation of a function germ, Lemma 3.2 and Proposition 3.3, we have the following classification result of Legendrian stable germs. For any map germ $f : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, we define the local ring of f by $Q(f) = \mathcal{E}_n / f^*(\mathfrak{M}_p) \mathcal{E}_n$.

Proposition 3.5 ^[7] Let $F, G : (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families. Suppose that Φ_F, Φ_G are Legendrian stable. Then the following conditions are equivalent.

- (1) $(W(\Phi_F), \mathbf{0})$ and $(W(\Phi_G), \mathbf{0})$ are diffeomorphic as germs.
- (2) Φ_F and Φ_G are Legendrian equivalent.
- (3) $Q(f)$ and $Q(g)$ are isomorphic as \mathbb{R} -algebras, where $f = F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$, $g = G|_{\mathbb{R}^k \times \{\mathbf{0}\}}$.

We now have tools for the study of the contact between Lorentzian surface and lightlike hyperplanes. Let $LP_{M,i}^\sigma : (U, (x_i, y_i)) \longrightarrow (LC_0, \mathbf{v}_i^\sigma)$ ($i = 1, 2$) be two lightcone pedal surface germs of Lorentzian surface germs $\mathbf{X}_i : (U, (x_i, y_i)) \longrightarrow (\mathbb{R}_2^4, p_i)$, where $\sigma = \pm$. We say that $LP_{M,1}^\sigma$ and $LP_{M,2}^\sigma$ are \mathcal{A} -equivalent if there exist diffeomorphism germs $\phi : (U, (x_1, y_1)) \longrightarrow (U, (x_2, y_2))$ and $\Phi : (LC_0, \mathbf{v}_1^\sigma) \longrightarrow (LC_0, \mathbf{v}_2^\sigma)$ such that $\Phi \circ LP_{M,1}^\sigma = LP_{M,2}^\sigma \circ \phi$. If the both of the regular sets of $LP_{M,i}^\sigma$ are dense in $(U, (x_i, y_i))$, it follows from proposition 3.5 that $LP_{M,1}^\sigma$ and $LP_{M,2}^\sigma$ are \mathcal{A} -equivalent if and only if the corresponding Legendrian lift germs are Legendrian equivalent. This condition is also equivalent to the condition that two generating families \tilde{H}_1 and \tilde{H}_2 are P - \mathcal{K} -equivalent by Theorem 3.4. Here, $\tilde{H}_i : (U \times LC_0, ((x_i, y_i), \mathbf{v}_i^\sigma)) \longrightarrow \mathbb{R}$ is the extended Lorentzian lightcone height function germ of \mathbf{X}_i .

On the other hand, we denote that $\tilde{h}_{i,v_i^\sigma}(u) = \tilde{H}_i(u, \mathbf{v}_i^\sigma)$, then we have $\tilde{h}_{i,v_i^\pm}(u) = \mathfrak{h}_{v_i^\pm} \circ \mathbf{X}_i(u)$. By Theorem 3.1, $K(\mathbf{X}_1(U), LHP(\mathbf{v}_1^\sigma, -1), \mathbf{v}_1^\sigma) = K(\mathbf{X}_2(U), LHP(\mathbf{v}_2^\sigma, -1), \mathbf{v}_2^\sigma)$ if and only if \tilde{h}_{1,v_1} and \tilde{h}_{1,v_2} are \mathcal{K} -equivalent. Therefore, we can apply the previous arguments to our situation. We denote $Q^\sigma(\mathbf{X}, (x_0, y_0))$ the local ring of the function germ $\tilde{h}_{v_0^\sigma} : (U, (x_0, y_0)) \longrightarrow \mathbb{R}$, where $v_0^\sigma = LP_M^\sigma(x_0, y_0)$. We remark that we can explicitly write the local ring as follows:

$$Q^\pm(\mathbf{X}, (x_0, y_0)) = \frac{C_{(x_0, y_0)}^\infty(U)}{\langle \langle \mathbf{X}(x, y), \mathbf{e}_1 \pm \mathbf{e}_2(x_0, y_0) \rangle - 1 \rangle_{C_{(x_0, y_0)}^\infty(U)}},$$

where $C_{(x_0, y_0)}^\infty(U)$ is the local ring of function germs at (x_0, y_0) with the unique maximal ideal $\mathfrak{M}_{(x_0, y_0)}(U)$.

Theorem 3.6 Let $\mathbf{X}_i : (U, (x_i, y_i)) \longrightarrow (\mathbb{R}_2^4, \mathbf{X}_i(x_i, y_i))$ ($i = 1, 2$) be an immersion germs with $\mathbf{X}(U) = M$ is a Lorentzian surface such that the corresponding Legendrian lift germs are Legendrian stable and $\sigma = \pm$. Then the following conditions are equivalent:

- (1) lightcone pedal surface germs $LP_{M,1}^\sigma$ and $LP_{M,2}^\sigma$ are \mathcal{A} -equivalent.
- (2) \tilde{H}_1 and \tilde{H}_2 are P - \mathcal{K} -equivalent.
- (3) \tilde{h}_{1,v_1} and \tilde{h}_{1,v_2} are \mathcal{K} -equivalent.
- (4) $K(\mathbf{X}_1(U), LHP(\mathbf{v}_1^\sigma, c_1^\sigma), \mathbf{v}_1^\sigma) = K(\mathbf{X}_2(U), LHP(\mathbf{v}_2^\sigma, c_2^\sigma), \mathbf{v}_2^\sigma)$
- (5) $Q^\sigma(\mathbf{X}_1, (x_1, y_1))$ and $Q^\sigma(\mathbf{X}_2, (x_2, y_2))$ are isomorphic as \mathbb{R} -algebras.

Proof. By the previous arguments (mainly by Theorem 3.1), it has been already shown that conditions (3) and (4) are equivalent. Other assertions follow from proposition 3.5. \square

Given an immersion germ $\mathbf{X} : (U, (x_0, y_0)) \longrightarrow (\mathbb{R}_2^4, \mathbf{X}(x_0, y_0))$ with $\mathbf{X}(U) = M$ is a Lorentzian surface, we call each set

$$(\mathbf{X}^{-1}(LHP(\mathbf{v}^\pm, c^\pm)), (x_0, y_0))$$

a *tangent lightlike hyperplane indicatrix germ* of \mathbf{X} , where $\mathbf{v}^\pm = \mathbf{e}_1 \pm \mathbf{e}_2(x_0, y_0)$ and $c^\pm = \langle \mathbf{X}(x_0, y_0), \mathbf{v}^\pm \rangle$. Moreover, by the above results, we can borrow some basic invariants from the singularity theory on function germs. We need \mathcal{K} -invariants for function germ. The local ring of a function germ is a complete \mathcal{K} -invariant for generic function germs. It is, however, not a numerical invariant. The \mathcal{K} -codimension (or, Tyurina number) of a function germ is a numerical \mathcal{K} -invariant of function germs [12]. We denote that

$$\text{L-ord}^\pm(\mathbf{X}, (x_0, y_0)) = \dim_{\mathbb{R}} \frac{C_{(x_0, y_0)}^\infty(U)}{\langle \tilde{h}_{\mathbf{v}_0^\pm}(x, y), \tilde{h}_{\mathbf{v}_0^\pm, x}(x, y), \tilde{h}_{\mathbf{v}_0^\pm, y}(x, y) \rangle}.$$

Usually $\text{L-ord}^\sigma(\mathbf{x}, u_0)$ is called *the \mathcal{K} -codimension of $\tilde{h}_{\mathbf{v}_0^\sigma}$* , where $\sigma = \pm$. However, we call it the *order of contact with the tangent lightlike hyperplane* at $\mathbf{X}(x_0, y_0)$. We also have the notion of corank of function germs.

$$\text{L-corank}^\sigma(\mathbf{X}, (x_0, y_0)) = 2 - \text{rank Hess}(\tilde{h}_{\mathbf{v}_0^\sigma}(x_0, y_0)),$$

where $\mathbf{v}_0^\pm = \mathbf{e}_1 \pm \mathbf{e}_2(x_0, y_0)$.

By proposition 2.1, $\mathbf{X}(x_0, y_0)$ is a L^σ -parabolic point if and only if $\text{L-corank}^\sigma(\mathbf{X}, (x_0, y_0)) \geq 1$. Moreover $\mathbf{X}(x_0, y_0)$ is a lightlike umbilic point if and only if $\text{L-corank}^\sigma(\mathbf{X}, (x_0, y_0)) = 2$.

On the other hand, a function germ $f : (\mathbb{R}^{n-1}, \mathbf{a}) \longrightarrow \mathbb{R}$ has the A_k -type singularity if f is \mathcal{K} -equivalent to the germ $\pm u_1^2 \pm \cdots \pm u_{n-2}^2 + u_{n-1}^{k+1}$. If $\text{L-corank}^\sigma(\mathbf{X}, (x_0, y_0)) = 1$, the extended Lorentzian lightcone height function $\tilde{h}_{\mathbf{v}_0^\sigma}$ has the A_k -type singularity at (x_0, y_0) in generic. In this case we have $\text{L-ord}^\sigma(\mathbf{x}, u_0) = k$. This number k is equal to the order of contact in the classical sense (cf., [4]). This is the reason why we call $\text{L-ord}^\sigma(\mathbf{X}, (x_0, y_0))$ the order of contact with the tangent lightlike hyperplane at $\mathbf{X}(x_0, y_0)$.

As a corollary of the theorem 3.6, we have the following result.

Corollary 3.7 Under the assumptions of Corollary 3.6, if the tangent lightlike hyperplane indicatrix germ $LHP_{M_1}^\sigma$ and $LHP_{M_2}^\sigma$ are \mathcal{A} -equivalent, then tangent lightcone indicatrix germs

$$(\mathbf{X}^{-1}(LHP(\mathbf{v}_1^\pm, c_1^\pm)), (x_1, y_1)) \quad \text{and} \quad (\mathbf{X}^{-1}(LHP(\mathbf{v}_2^\pm, c_2^\pm)), (x_2, y_2))$$

are diffeomorphic as set germs.

Proof. Notice that the tangent lightlike hyperplane indicatrix germ of \mathbf{X}_i is the zero level set of h_{i, λ_i} . Since \mathcal{K} -equivalence among function germs preserves the zero-level sets of function germs, the assertion follows from theorem 3.6. \square

4 Classification of singularities of $S_t^1 \times S_s^2$ -valued lightcone Gauss maps and lightcone pedalsurfaces

In this section we consider generic singularities of $S_t^1 \times S_s^2$ -valued lightcone Gauss maps and lightcone pedal surfaces. We consider the space of Lorentzian embeddings $\text{Emb}_L(U, \mathbb{R}_2^4)$ with Whitney C^∞ -topology, where $U \subset \mathbb{R}^2$ is an open subset. We have the following theorem.

Theorem 4.1 There exists an open dense subset $\mathcal{O} \subset \text{Emb}_L(U, \mathbb{R}_2^4)$ such that for any $\mathbf{X} \in \mathcal{O}$, the following conditions hold:

(1) Each lightlike parabolic set $\mathcal{K}_l(1, \sigma 1)^{-1}(0)$ is a regular curve. We call such a curve *the lightlike parabolic curve*.

(2) The lightcone pedal surface LP_M^σ along the lightlike parabolic curve is the cuspidaledge except isolated points. At these points LP_M^σ is the swallowtail.

Here, $\sigma = \pm$ and a map germ $f : (\mathbb{R}^2, \mathbf{a}) \longrightarrow (\mathbb{R}^3, \mathbf{b})$ is called a *cuspidaledge* if it is \mathcal{A} -equivalent to the germ (u_1, u_2^2, u_2^3) (cf., Fig. 1) and a *swallowtail* if it is \mathcal{A} -equivalent to the germ $(3u_1^4 + u_1^2 u_2, 4u_1^3 + 2u_1 u_2, u_2)$.

For the proof of Theorem 4.1, we consider the function $\mathcal{H} : \mathbb{R}_2^4 \times LC_0 \longrightarrow \mathbb{R}$ which is given in §3. We claim that \mathcal{H}_v is a submersion for any $\mathbf{v} \in LC_0$, where $\mathcal{H}_v(\mathbf{x}) = \mathcal{H}(\mathbf{x}, \mathbf{v})$. For any $\mathbf{X} \in \text{Emb}_L(U, \mathbb{R}_2^4)$, we have $\tilde{H} = \mathcal{H} \circ (\mathbf{X} \times id_{LC_0})$. We also have the ℓ -jet extension

$$j_1^\ell \tilde{H} : U \times LC_0 \longrightarrow J^\ell(U, \mathbb{R})$$

defined by $j_1^\ell \tilde{H}((x, y), \mathbf{v}) = j_1^\ell \tilde{h}_v(x, y)$. We consider the trivialization $J^\ell(U, \mathbb{R}) \equiv U \times \mathbb{R} \times J^\ell(2, 1)$. For any submanifold $Q \subset J^\ell(2, 1)$, we denote that $\tilde{Q} = U \times \{0\} \times Q$. Then we have the following proposition as a corollary of Lemma 6 in Wassermann [17]. (See also Montaldi [14] and Looijenga [11]).

Proposition 4.2 Let Q be a submanifold of $J^\ell(2, 1)$. Then the set

$$T_Q = \{\mathbf{x} \in \text{Emb}_L(U, \mathbb{R}_2^4) \mid j_1^\ell H \text{ is transversal to } \tilde{Q}\}$$

is a residual subset of $\text{Emb}_s(U, \mathbb{R}_2^4)$. If Q is a closed subset, then T_Q is open.

If we consider \mathcal{K} -orbits in $J^\ell(2, 1)$, we obtain the proof of Theorem 4.1, so that we omit the detailed discussion. The assertion of Theorem 4.1 can be interpreted that the Legendrian lift of the lightcone pedal surface LP_M^\pm of $\mathbf{X} \in \mathcal{O}$ is Legendrian stable at each point. Since the Legendrian lift is the Legendrian covering of the Lagrangian lift of LG_M^\pm , it has been known that the corresponding singularities of LG_M^\pm are folds or cusps [1]. Hence, we have the following corollary.

Corollary 4.3 Let $\mathcal{O} \subset \text{Emb}_L(U, \mathbb{R}_2^4)$ be the same open dense subset as in Theorem 4.1. For any $\mathbf{X} \in \mathcal{O}$, the followings hold:

(1) A lightlike parabolic point $(x_0, y_0) \in U$ is a fold of the $S_t^1 \times S_s^2$ -valued lightcone Gauss map LG_M^σ if and only if it is a cuspidaledge of the lightcone pedal surface LP_M^σ .

(2) A lightlike parabolic point $(x_0, y_0) \in U$ is a cusp of the $S_t^1 \times S_s^2$ -valued lightcone Gauss map LG_M^σ if and only if it is a swallowtail of the lightcone pedal surface LP_M^σ .

Here, a map germ $f : (\mathbb{R}^2, \mathbf{a}) \longrightarrow (\mathbb{R}^2, \mathbf{b})$ is called a *fold* if it is \mathcal{A} -equivalent to the germ (u_1, u_2^2) and a *cuspidaledge* if it is \mathcal{A} -equivalent to the germ $(u_1, u_2^3 + u_1 u_2)$.

Following the terminology of Whitney [18], we say that a surface $\mathbf{X} : U \longrightarrow \mathbb{R}_2^4$ has *the excellent lightcone pedal surface* LP_M^σ if the Legendrian lift of LP_M^σ is a stable Legendrian immersion at each point. In this case, the lightcone pedal surface LP_M^σ has only cuspidaledges and swallowtails as singularities. Proposition 4.1 asserts that a Lorentzian surface with the excellent lightcone pedal surface is generic in the space of all Lorentzian surface in \mathbb{R}_2^4 . We now consider the geometric meanings of cuspidaledges and swallowtails of the lightcone pedal surface. We have the following results analogous to the results of Banchoff et al [2].

Theorem 4.4 Let $LP_M^\sigma : (U, (x_0, y_0)) \longrightarrow (\mathbb{R}_2^4, p_0)$ be the excellent lightcone pedal surface of a Lorentzian surface \mathbf{X} and $\tilde{h}_{v_0^\sigma} : (U, (x_0, y_0)) \longrightarrow \mathbb{R}$ be the extended lightcone height function germ at $v_0^\pm = \mathbf{e}_1 \pm \mathbf{e}_2(p_0)$, where $\sigma = \pm$. Then we have the following:

- (1) (x_0, y_0) is a lightlike parabolic point of \mathbf{X} if and only if $\text{L-corank}^\sigma(\mathbf{X}, (x_0, y_0)) = 1$.
- (2) If (x_0, y_0) is a lightlike parabolic point of \mathbf{X} , then $\tilde{h}_{v_0^\sigma}$ has the A_k -type singularity for $k = 2, 3$.
- (3) Suppose that (x_0, y_0) is a lightlike parabolic point of \mathbf{X} . Then the following conditions are equivalent:

- (a) LP_M^σ has a cuspidal edge at (x_0, y_0)
- (b) $\tilde{h}_{v_0^\sigma}$ has the A_2 -type singularity.
- (c) $\text{L-ord}^\sigma(\mathbf{X}, (x_0, y_0)) = 2$.
- (d) The tangent lightlike hyperplane indicatrix is a ordinary cusp, where a curve $C \subset \mathbb{R}^2$ is called a *ordinary cusp* if it is diffeomorphic to the curve given by $\{(u_1, u_2) \mid u_1^2 - u_2^3 = 0\}$.
- (e) For each $\varepsilon > 0$, there exist two distinct points $(x_i, y_i) \in U$ ($i = 1, 2$) such that

$$\|(x_0, y_0) - (x_i, y_i)\| < \varepsilon$$

for $i = 1, 2$, both of (x_i, y_i) are not lightlike parabolic points and the tangent lightlike hyperplanes to $M = \mathbf{x}(U)$ at (x_i, y_i) are parallel.

- (4) Suppose that (x_0, y_0) is a lightlike parabolic point of \mathbf{X} . Then the following conditions are equivalent:

- (a) LP_M^σ has a swallowtail at (x_0, y_0)
- (b) $\tilde{h}_{v_0^\sigma}$ has the A_3 -type singularity.
- (c) $\text{L-ord}^\sigma(\mathbf{X}, (x_0, y_0)) = 3$.
- (d) The tangent lightlike hyperplane indicatrix is a point or a tachnodal, where a curve $C \subset \mathbb{R}^2$ is called a *tachnodal* if it is diffeomorphic to the curve given by $\{(u_1, u_2) \mid u_1^2 - u_2^4 = 0\}$.
- (e) For each $\varepsilon > 0$, there exist three distinct points $(x_i, y_i) \in U$ ($i = 1, 2, 3$) such that

$$\|(x_0, y_0) - (x_i, y_i)\| < \varepsilon$$

for $i = 1, 2, 3$ and the tangent lightlike hyperplanes to $M = \mathbf{x}(U)$ at (x_i, y_i) are parallel.

- (f) For each $\varepsilon > 0$, there exist two distinct points $(x_i, y_i) \in U$ ($i = 1, 2$) such that

$$\|(x_0, y_0) - (x_i, y_i)\| < \varepsilon$$

for $i = 1, 2$ and the tangent lightlike hyperplanes to $M = \mathbf{x}(U)$ at (x_i, y_i) are equal.

Proof. We have shown that (x_0, y_0) is a lightlike parabolic point if and only if

$$\text{L-corank}^\sigma(\mathbf{X}, (x_0, y_0)) \geq 1.$$

We have $\text{L-corank}^\sigma(\mathbf{X}, (x_0, y_0)) \leq 2$. Since the extended lightcone height function germ $\tilde{H} : (U \times LC_0, ((x_0, y_0), v_0)) \longrightarrow \mathbb{R}$ can be considered as a generating family of the Legendrian lift of LP_M^σ , $\tilde{h}_{v_0^\sigma}$ has only the A_k -type singularities ($k = 1, 2, 3$). This means that the corank of the Hessian matrix of $\tilde{h}_{v_0^\sigma}$ at a lightlike parabolic point is 1. The assertion (2) also follows. By the same reason, the conditions (3);(a),(b),(c) (respectively, (4); (a),(b),(c)) are equivalent. If the height function germ $\tilde{h}_{v_0^\sigma}$ has the A_2 -type singularity, then it is \mathcal{K} -equivalent to the germ $\pm u_1^2 + u_2^3$.

Since the \mathcal{K} -equivalence preserve the diffeomorphism type of zero level sets, the tangent lightlike hyperplane indicatrix is diffeomorphic to the curve given by $\pm u_1^2 + u_2^3 = 0$. This is the ordinary cusp. The normal form for the A_3 -type singularity is given by $\pm u_1^2 + u_2^4$, so that the tangent lightlike hyperplane indicatrix is diffeomorphic to the curve $\pm u_1^2 + u_2^4 = 0$. This means that the condition (3),(d) (respectively, (4),(d)) is also equivalent to the other conditions.

Suppose that (x_0, y_0) is a lightlike parabolic point, then the $S_t^1 \times S_s^2$ -valued lightcone Gauss map has only folds or cusps. If the point (x_0, y_0) is a fold point, there is a neighbourhood of (x_0, y_0) on which the $S_t^1 \times S_s^2$ -valued lightcone Gauss map is 2 to 1 except at the lightlike parabolic curve (i.e, fold curve). By Lemma 3.2, the condition (3), (e) is satisfied. If the point (x_0, y_0) is a cusp, the critical value set is an ordinary cusp. By the normal form, we can understand that the $S_t^1 \times S_s^2$ -valued lightcone Gauss map is 3 to 1 inside region of the critical vale. Moreover, the point (x_0, y_0) is in the closure of the region. This means that the condition (4),(e) holds. We can also observe that near by a cusp point, there are 2 to 1 points which approach to (x_0, y_0) . However, one of those points are always lightlike parabolic points. Since other singularities do not appear for in this case, so that the condition (3),(e) (respectively, (4),(e)) characterizes a fold (respectively, a cusp).

If we consider the lightcone pedal surface in stead of the lightcone Gauss map, the only singularities are cuspidaledges or swallowtails. For the swallowtail point (x_0, y_0) , there is a self intersection curve approaching to (x_0, y_0) . On this curve, there are two distinct point (x_i, y_i) ($i = 1, 2$) such that $LP_M^\sigma(x_1, y_1) = LP_M^\sigma(x_2, y_2)$. By Lemma 3.2, this means that tangent lightlike hyperplane to $M = \mathbf{x}(U)$ at (x_i, y_i) are equal. Since there are no other singularities in this case, the condition (4),(f) characterize a swallowtail point of LP_M^σ . This completes the proof. \square

References

- [1] V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps vol. I*, Birkhäuser (1986)
- [2] T. Banchoff, T. Gaffney and C. McCrory, *Cusps of Gauss Mappings*, Research Notes in Mathematics **55** Pitman, London (1982)
- [3] D. Bleeker and L. Wilson, *Stability of Gauss maps*, Illinois J. Math. **22** (1978), 279–289.
- [4] J. W. Bruce and P. J. Giblin, *Curves and singularities (second edition)*, Cambridge University press, (1992)
- [5] J. Ehlers and E. T. Newman, *The theory of caustics and wave front singularities with physical applications*, Journal of Mathematical Physics, **41** (2000), 3344–3378.
- [6] S.Izumiya and D-H.Pei, The $S_t^1 \times S_s^2$ - valued lightcone Gauss map of a Lorentzian 3-submanifold in semi-Euclidean5-space, Hokkaido University Technical Report Series in Mathematics,78 (2003)163-166.
- [7] S.Izumiya, D-H.Pei and T.Sano, *Singularities of hyperbolic Gauss maps*, Proc.London.Math.Soc. (3) **86** (2003), 485–512
- [8] S. Izumiya, D-H. Pei and M. C. Romero Fuster, The lightcone Gauss map of a spacelike surfaces in Minkowski 4-space, Asian Journal of Mathematics, 8(2004) 511-530.

- [9] S. Izumiya, M. Kossowski, D-H. Pei and M. C. Romero Fuster, Singularities of lightlike hypersurfaces in Minkowski 4-space, *Tohoku Mathematical Journal*, **58** (2006), 71-88.
- [10] J. A. Little, *On singularities of submanifolds of high dimensional Euclidean space*, *Annali Mat.Pura et Appl.(ser.4A)*, **83** (1969), 261-336.
- [11] E. J. N. Looijenga, *Structural stability of smooth families of C^∞ -functions*, Thesis, Univ. Amsterdam (1974)
- [12] J. Martinet, *Singularities of Smooth Functions and Maps*, London Math. Soc. Lecture Note Series, Cambridge Univ. Press, **58** (1982)
- [13] J. A. Montaldi, *On contact between submanifolds*, *Michigan Math. J.*, **33** (1986), 81-85
- [14] J. A. Montaldi, *On generic composites of maps*. *Bull. London Math. Soc.*, **23** (1991), 81-85
- [15] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York (1983)
- [16] D-H. Pei, *Singularities of $\mathbb{R}P^2$ -valued Gauss maps of surfaces in Minkowski 3-space*, *Hokkaido Mathematical Journal*, **28**, (1999), 97-115.
- [17] G. Wassermann, *Stability of Caustics*, *Math. Ann.*, **2210** (1975), 443-50.
- [18] H. Whitney, *On singularities of mappings of Euclidean spaces I*, *Ann. of Math.*, **62** (1955), 374-410.
- [19] V. M. Zakalyukin, *Lagrangian and Legendrian singularities*. *Funct. Anal. Appl.*, **10** (1976), 23-31.
- [20] V. M. Zakalyukin, *Reconstructions of fronts and caustics depending one parameter and versality of mappings*. *J. Sov. Math.*, **27** (1984), 2713-2735.

Donghe Pei, School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, P.R.China

e-mail: peidh340@nenu.edu.cn

Jianguo Sun, School of Mathematics ,China University of petroleum, Dongying 257061, P.R.China

e-mail: sunjg616@163.com

Qi Wang, School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, P.R.China